Fluid models of plasma

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Introduction

Fluid model: describe plasma in terms of macroscopic (averaged) quantities:

–Average velocity

–Charge and current density

–Temperature, pressure

Governed by equations of state, together with Maxwell's equations

Basics

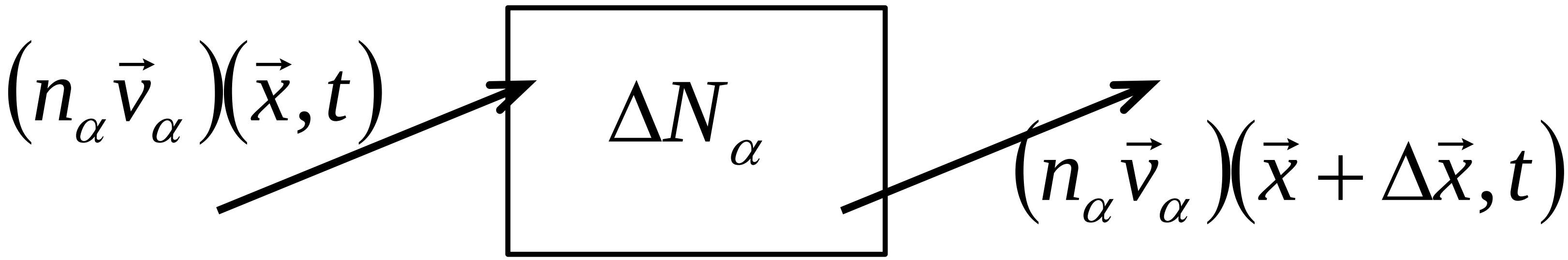
Consider a plasma consisting of electrons and one or more ion species

- Particle charge, mass: $Z_\alpha e$ and m_α
- Species density, velocity: $n_\alpha(x,t)$, $v_\alpha(x,t)$
- Fully ionised
- (Almost) all collisions are elastic
- No ionisation, recombination, radiation

Number of particles is conserved for each species

Particle conservation

Consider a small volume of plasma:



Change in number of particles =

Particle flux into volume —

Particle flux out of this volume

Particle conservation

In one dimension:

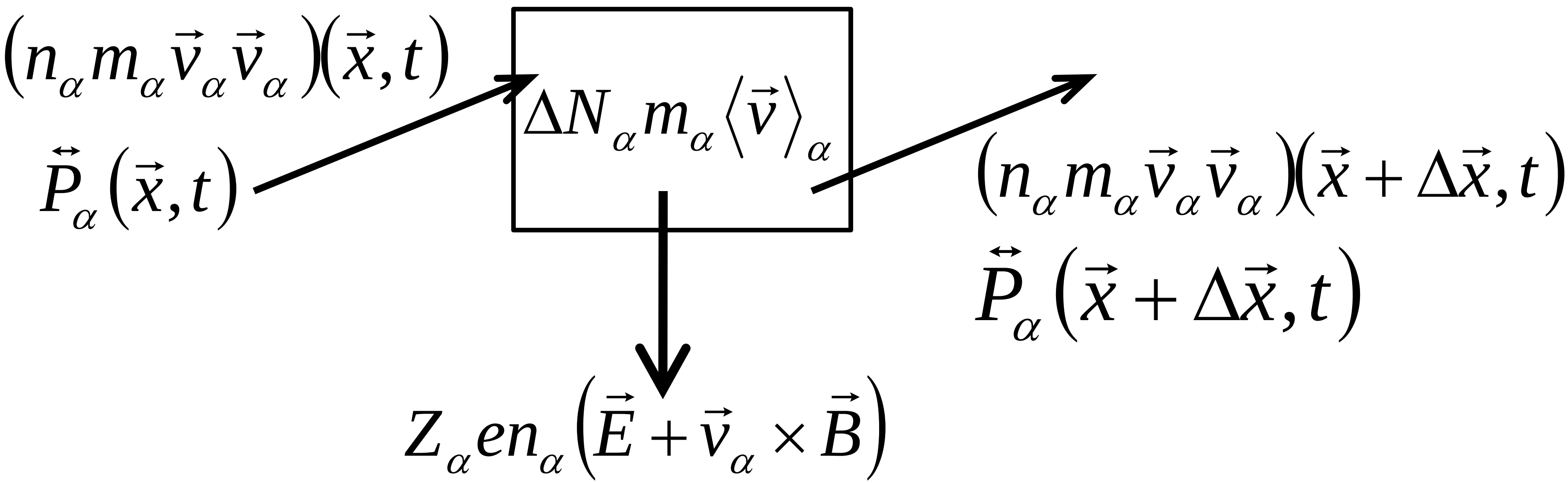
$$\Delta N_\alpha \approx \frac{\partial n_\alpha}{\partial t}\Delta x \Delta t = \left[(n_\alpha v_\alpha)(x,t) - (n_\alpha v_\alpha)(x+\Delta x, t)\right]\Delta t \approx -\frac{\partial (n_\alpha v_\alpha)}{\partial x}\Delta x \Delta t$$

or: $\dfrac{\partial n_\alpha}{\partial t} + \dfrac{\partial (n_\alpha v_\alpha)}{\partial x} = 0$

In three dimensions: $\dfrac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{v}_\alpha) = 0$

Continuity equation

Momentum conservation



Change in momentum = Flux in — flux out + pressure left — pressure right + volume forces

Intermezzo: tensors

Particle flux: simple vector, one bit of information: direction of flux

Momentum flux: two bits of information: direction of momentum that fluxes, and direction of flux

Cannot be expressed as vector!

Tensors 2

Momentum flux is tensor (matrix):

$$\left(m\vec{v}\vec{v}\right)_{ij} \equiv m v_i v_j$$

First index: which momentum component fluxes

Second index: component of direction of flux

Tensors 3

Pressure is also a tensor: need to consider direction of pressure force, and direction of normal vector on surface!

$$\overleftrightarrow{P} \equiv \left(P\right)_{ij}$$

First index: component of force

Second index: component of surface normal

Simple tensor rules

Divergence of tensor is vector:

$$\left(\nabla \cdot \overleftrightarrow{P}\right)_i \equiv \sum_j \frac{\partial P_{ij}}{\partial x_j}$$

Gradient of vector is tensor:

$$\left(\nabla \vec{b}\right)_{ij} \equiv \frac{\partial b_i}{\partial x_j}; \quad \nabla \vec{b} \neq \nabla \cdot \vec{b}$$

Two vectors can make a tensor:

$$\left(\vec{v}\vec{w}\right)_{ij} = v_i w_j$$

Momentum balance

$$\frac{\partial}{\partial t}\left(n_\alpha m_\alpha \vec{v}_\alpha\right) + \nabla \cdot \left(n_\alpha m_\alpha \vec{v}_\alpha \vec{v}_\alpha\right) + \nabla \cdot \vec{P}_\alpha =$$

$$= Z_\alpha e n_\alpha \left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right) + \vec{R}_\alpha$$

1. Change in momentum density
2. Divergence of momentum flux
3. Divergence of pressure
4. Volume forces, e.g. EM forces or gravity
5. Collisions with other species: $\sum_\alpha \vec{R}_\alpha = 0$

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Example 1: static "flow"

Fluid is static, v=0, pressure forces balance volume forces:

$$\nabla \cdot \vec{P}_\alpha = -m_\alpha n_\alpha g\hat{z} + Z_\alpha e n_\alpha \left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right) + \vec{R}_\alpha$$

Added gravity force -ρ*g

Consequence: P = ρ*g*h for constant gravity and fluid/plasma depth h

Example 2: steady flow

Drop all time derivatives, use gravity for the volume force:

$$\nabla \cdot \left(n_\alpha \vec{v}_\alpha \right) = 0 \qquad \nabla \cdot \left(n_\alpha m_\alpha \vec{v}_\alpha \vec{v}_\alpha \right) + \nabla \cdot \vec{P}_\alpha = -n_\alpha m_\alpha g \hat{z}$$

Use isotropic P and sum over tensor diagonals: Bernouiilli's equation!

$$P + \frac{1}{2} n_\alpha m_\alpha v_\alpha^2 + n_\alpha m_\alpha g h = C$$

Closing the system

The continuity equation together with the momentum balance do not constitute a closed system

Need to add equations of state for pressure P and collision terms R

Think of: Ohm's law, expressions for adiabatic or isothermal compression

Pressure

Write the pressure tensor as follows:

$$\overleftrightarrow{P} = p\overleftrightarrow{I} + \overleftrightarrow{\Pi}$$

Fast processes in collision-poor plasma are adiabatic:

$$p = Cn^{5/3}$$

Slow processes in collision-rich plasma are isothermal:

$$p = Cn$$

Distribution function

For each particle species, a distribution function f_α is introduced

Number of particles in volume element:

$$f_\alpha(\vec{x},\vec{v},t)d^3\vec{x}d^3\vec{v}$$

This number is changed by collisions:

$$\frac{d}{dt}\left[f_\alpha(\vec{x},\vec{v},t)d^3\vec{x}d^3\vec{v}\right]=C_\alpha d^3\vec{x}d^3\vec{v}$$

Boltzmann equation

We have:

$$\frac{d}{dt}\left[f_\alpha(\vec{x},\vec{v},t)d^3\vec{x}d^3\vec{v}\right]=\left[\frac{df_\alpha}{dt}+f_\alpha\left(\nabla_{\vec{x}}\cdot\dot{\vec{x}}+\nabla_{\vec{v}}\cdot\dot{\vec{v}}\right)\right]d^3\vec{x}d^3\vec{v}$$

but also:

$$\dot{\vec{x}}=\vec{v};\quad \dot{\vec{v}}=\frac{Z_\alpha e}{m_\alpha}\left(\vec{E}+\vec{v}\times\vec{B}\right);\quad \frac{\partial\dot{x}_i}{\partial x_i}=\frac{\partial\dot{v}_i}{\partial v_i}=0$$

Boltzmann equation

Boltzmann equation (Ludwig Boltzmann, 1844-1906)

$$\frac{d}{dt}\left[f_\alpha\left(\vec{x},\vec{v},t\right)\right] \equiv \frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla_{\vec{x}} f_\alpha + \frac{Z_\alpha e}{m_\alpha}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_{\vec{v}} f_\alpha = C_\alpha$$

Collisional term: $C_\alpha = \sum_\beta C_{\alpha\beta}$

No collisions: Vlasov equation (Anatoly Vlasov, 1908-1975)

Charge, current

Particle density: $n_\alpha(\vec{x},t) \equiv \int f_\alpha(\vec{x},\vec{v},t)\, d^3\vec{v}$

Average velocity: $n_\alpha \vec{u}_\alpha(\vec{x},t) \equiv \int f_\alpha(\vec{x},\vec{v},t)\, \vec{v}\, d^3\vec{v}$

Charge density: $\rho(\vec{x},t) \equiv \sum_\alpha Z_\alpha e n_\alpha$

Current density: $\vec{j}(\vec{x},t) \equiv \sum_\alpha Z_\alpha e n_\alpha \vec{u}_\alpha$

Sources for Maxwell's equations

The Vlasov-Maxwell system is closed

The Vlasov-Boltzmann system needs an additional collision model

Moment equations

Vlasov equation hard to tackle analytically

Solution: use moments:

–Multiply by some power of v

–Integrate over all v

–Use: $\lim_{v\to\infty} v^n f_\alpha(\vec{x},\vec{v},t) = 0$

–Define: $n_\alpha \langle \Psi \rangle_\alpha \equiv \int \Psi(\vec{v}) f_\alpha(\vec{x},\vec{v},t) d^3\vec{v}$

for any function Ψ

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Particle conservation

Use $\Psi = 1$ to obtain "zeroth" order moment equation:

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha \vec{u}_\alpha \right) = 0$$

Note that this equation contains the first order moment u.

Momentum conservation

Use $\Psi = m_\alpha \vec{v}$ to obtain first order moment equation:

$$\frac{\partial}{\partial t}\left(n_\alpha m_\alpha \vec{u}_\alpha\right) + \nabla \cdot \left(n_\alpha m_\alpha \left\langle \vec{v}\vec{v} \right\rangle_\alpha\right) - Z_\alpha e n_\alpha \left(\vec{E} + \vec{u}_\alpha \times \vec{B}\right) = \sum_\beta \int m_\alpha \vec{v} C_{\alpha\beta} d^3\vec{v}$$

Note that this equation contains the second order moment <vv>.

Energy conservation

Use $\Psi = m_\alpha v^2 / 2$ to obtain second order moment equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2} n_\alpha m_\alpha \left\langle v^2 \right\rangle_\alpha \right) + \nabla \cdot \left(\frac{1}{2} n_\alpha m_\alpha \left\langle v^2 \vec{v} \right\rangle_\alpha \right) - Z_\alpha e n_\alpha \vec{u}_\alpha \cdot \vec{E} = \sum_\beta \int \frac{1}{2} m_\alpha v^2 C_{\alpha\beta} d^3\vec{v}$$

Note that this equation contains the third order moment $\langle v^2 \mathbf{v} \rangle$.

Closing the system

Moment equation of order n contains moment of order n+1.

System is never closed!

Need to provide equation of state for highest order moment (or just drop it):

–Neglect heat flux (fast processes)

–Prescribe pressure

–Add viscosity model

Grooming the equations

Relative velocity: $\vec{v}'_\alpha \equiv \vec{v} - \vec{u}_\alpha$

Pressure tensor: $\overleftrightarrow{P}_\alpha \equiv m_\alpha n_\alpha \langle \vec{v}'_\alpha \vec{v}'_\alpha \rangle = p_\alpha \overleftrightarrow{I} + \overleftrightarrow{\Pi}_\alpha$

Isotropic pressure: $p_\alpha \equiv \frac{1}{3} m_\alpha \int v'^2_\alpha f_\alpha d^3\vec{v} = \frac{1}{3} \sum_i P_{\alpha ii}$

Kinetic temperature: $kT_\alpha \equiv p_\alpha / n_\alpha$

Heat flux: $\vec{q}_\alpha \equiv \frac{1}{2} n_\alpha m_\alpha \langle v'^2_\alpha \vec{v}_\alpha \rangle$

Collisional exchanges:

$$\vec{R}_\alpha \equiv \sum_\beta \int m_\alpha \vec{v}'_\alpha C_{\alpha\beta} d^3\vec{v} \qquad Q_\alpha \equiv \sum_\beta \int \frac{1}{2} m_\alpha v'^2_\alpha C_{\alpha\beta} d^3\vec{v}$$

$$\sum_\alpha \vec{R}_\alpha = \vec{0} \qquad \sum_\alpha \left(Q_\alpha + \vec{u}_\alpha \cdot \vec{R}_\alpha \right)$$

Momentum revisited

Insert "physical" quantities and subtract u*(continuity equation):

$$mn\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right)=-\nabla p-\nabla\cdot\overleftrightarrow{\Pi}+Zen\left(\vec{E}+\vec{u}\times\vec{B}\right)+\vec{R}$$

Flow equation of electron/ion fluid

Left: advection

Right: pressure, viscosity, Lorentz force, momentum exchange between species

Energy revisited

Insert "physical" quantities and subtract u^2*(continuity equation) and u·(momentum equation):

$$\frac{3}{2}\frac{\partial p}{\partial t}+\frac{3}{2}\nabla\cdot\left(p\vec{u}\right)+p\nabla\cdot\vec{u}=-\nabla\cdot\vec{q}-\overleftrightarrow{\Pi}:\nabla\vec{u}+Q$$

$$\overleftrightarrow{A}:\overleftrightarrow{B}\equiv\sum_{i,j}A_{ij}B_{ij}$$

Left: Internal energy, work

Right: heat flux, viscosity, heat exchange between species

Example: adiabatic compression

Need to provide expressions for viscosity, heat flux, friction, heat exchange between species, to close system.

For adiabatic compression:

$$\overleftrightarrow{\Pi} = \overleftrightarrow{0} \quad \vec{q} = \vec{0} \quad Q = 0$$

$$\frac{3}{2}\left(\frac{\partial p}{\partial t} + \nabla \cdot \left(p\vec{u}\right)\right) + p\nabla \cdot \vec{u} = 0$$

$$d\left(pn^{-5/3}\right)/dt = 0$$

Recap of all components

Continuity: $$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha \vec{v}_\alpha \right) = 0$$

Momentum: $$\frac{\partial}{\partial t}\left(n_\alpha m_\alpha \vec{v}_\alpha \right) + \nabla \cdot \left(n_\alpha m_\alpha \vec{v}_\alpha \vec{v}_\alpha \right) + \nabla \cdot \vec{P}_\alpha =$$

$$= Z_\alpha e n_\alpha \left(\vec{E} + \vec{v}_\alpha \times \vec{B} \right) + \vec{R}_\alpha$$

Collisions: $$\vec{R}_e = -\vec{R}_i = -\frac{m_e n_e}{\tau_e}\left(\vec{v}_e - \vec{v}_i \right) = \frac{m_e}{e\tau_e}\vec{J}$$

Pressure: $$\overleftrightarrow{P} = p\overleftrightarrow{I} + \overleftrightarrow{\Pi}$$

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Pressure: $\overleftrightarrow{P} = p\overleftrightarrow{I} + \overleftrightarrow{\Pi}$

$$p_{\mathrm{ad}} = Cn^{5/3} \qquad\qquad p_{\mathrm{iso}} = Cn$$

$$\Pi_{ij} = -\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}\nu\nabla\cdot\vec{v}$$

Maxwell’s equations for E and B complete the system

Radiation in plasma

Radiation plays a central role in high-density plasma:

- Transfers energy from hot to cold regions
- Radiation often more effective for transporting energy than thermal conduction
- Connects time evolution of distant plasma regions
- In laser-produced plasma: decreases laser driving pressure and increases “preheat”

Radiation: basics

$$I\left(\vec{r},\nu,\vec{\Omega},t\right)=ch\nu f\left(\vec{r},\nu,\vec{\Omega},t\right)$$

$$dE = I\left(\vec{r},\nu,\vec{\Omega},t\right)\cos\theta d\nu d\vec{\Omega}dtd\sigma$$

For homogeneous, isotropic radiation in TE: Planck formula

$$I(\nu)=B^{P}(\nu)=\frac{2h\nu^{3}}{c^{2}}\frac{1}{\exp(h\nu/kT)-1}$$

Energy density: $$U = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega} I \rightarrow \frac{4\sigma}{c}T^4$$

Flux: $$\vec{F} = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega} I \rightarrow 0$$

Pressure: $$\overleftrightarrow{P} = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega}\vec{\Omega} I \quad \left\langle \overleftrightarrow{P} \right\rangle = \frac{1}{3}\mathrm{Tr}\left(\overleftrightarrow{P}\right) = \frac{1}{3}U$$

Radiation transfer

- streaming in/out $\frac{\partial}{\partial x_i}(x_i, f)$
- absorption $c\alpha_a(\vec{r},\nu,t)f$
- out-scattering $\Sigma_{so} = c\alpha_s(\vec{r},\nu,t)f$
- in-scattering Σ_{si}
- emission $q(\vec{r},\nu,t) = j(\vec{r},\nu,t)/h\nu$

Transfer equation:

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\nabla I(\nu,\vec{\Omega}) = j(\nu) - [\alpha_a(\nu) + \alpha_s]I(\nu,\vec{\Omega}) + h\nu\Sigma_{si}$$

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Quantum effects

Photons are bosons: emission and scattering are proportional to the number of photons in the final state – induced processes

Quantum and classical probability are related via P' = P(1+n) where n is the number of photons in the unit cell of phase space:

$$n = \int d\vec{r} \int_{\Delta} d\nu \int d\vec{\Omega} f\left(\vec{r}, \nu, \vec{\Omega}, t\right) = \frac{c^2}{2h\nu^3} I\left(\vec{r}, \nu, \vec{\Omega}, t\right)$$

Transfer equation becomes:

$$\frac{1}{c}\frac{\partial I\left(\nu, \vec{\Omega}\right)}{\partial t} + \vec{\Omega} \cdot \nabla I\left(\nu, \vec{\Omega}\right) = j(\nu)\left[1 + \frac{c^2}{2h\nu^3} I\right] - \alpha_a(\nu) I\left(\nu, \vec{\Omega}\right) + \alpha_s\left[1 + \frac{c^2}{2h\nu^3} I\right] I\left(\nu, \vec{\Omega}\right) + h\nu\Sigma_{si}$$

Neglect scattering

Drop scattering terms but retain quantum effects (needed to recover the Planck equilibrium distribution):

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t}+\vec{\Omega}\cdot\nabla I(\nu,\vec{\Omega})= j(\nu)\left[1+\frac{c^2}{2h\nu^3}I\right]-\alpha_a(\nu)I(\nu,\vec{\Omega})$$

In TE, absorption and emission are connected via Kirchhoff's law:

$$j(\nu)=\alpha_a^{'}(\nu)B(\nu) \quad \alpha_a(\nu)=\alpha_a^{'}(\nu)\left[1+c^2B(\nu)/2h\nu^3\right]$$

Use the Planck function, $B^p(v)$ to obtain

$$\alpha_a^{'}(\nu)=\alpha_a(\nu)\left[1-\exp(-h\nu/kT)\right]$$

The exponential factor decreases the absorption due to stimulated emission

Summary

Fluid models are used to describe macroscopic plasma behaviour

Equations of state needed to close system of equations

These may be derived from an analysis of microscopic processes

Radiation models needed for dense plasma fluid models; see also lectures by Profs. T. Bell and S. Rose